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Lower Semicontinuity, Almost Lower Semicontinuity, and Continuous Selections for Set-Valued Mappings

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For a set-valued mapping, the relationships between lower semicontinuity, almost lower semicontinuity, and the existence of various kinds of continuous selections for the mapping are explored. For the spaces $C_0(T)$ and L_1, intrinsic characterizations are given of the one-dimensional subspaces whose metric projections admit continuous selections. © 1988 Academic Press, Inc.

1. INTRODUCTION

Let X be a paracompact space, Y a normed linear space, 2^Y the collection of all nonempty subsets of Y, and $\mathscr{C}(Y)$ the collection of all nonempty, closed, and convex subsets of Y. The ε-neighborhood of a set $A \in 2^Y$ is defined by

$$B_\varepsilon(A) := \{y \in Y \mid d(y, A) < \varepsilon\}, \qquad \text{where} \quad d(y, A) := \inf\{\|y-a\| \mid a \in A\}.$$

A function $F: X \to 2^Y$ is called a set-valued (or multivalued) mapping

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from X into Y. F is called *lower semicontinuous* (l.s.c.) at a point $x_0 \in X$ if whenever W is an open set in Y with $F(x_0) \cap W \neq \emptyset$, there exists a neighborhood U of x_0 such that $F(x) \cap W \neq \emptyset$ for every $x \in U$. F is called *almost lower semicontinuous* (a.l.s.c.) at x_0 if for each $\varepsilon > 0$, there exists a neighborhood U of x_0 such that

$$\bigcap_{x \in U} B_\varepsilon(F(x)) \neq \emptyset.$$

F is called *lower semicontinuous* (l.s.c.) [resp. *almost lower semicontinuous* (a.l.s.c.)] if it is l.s.c. (resp. a.l.s.c.) at each point of X.

A *selection* (resp. *ε-approximate selection*) for F is a function $f\colon X \to Y$ such that $f(x) \in F(x)$ (resp. $f(x) \in B_\varepsilon(F(x))$) for every x in X. Observe that every selection is an ε-approximate selection, but the converse is false in general.

The important and well-known selection theorem of Michael can be stated as follows.

1.1. Theorem (Michael [11]). *If Y is complete and $F\colon X \to \mathscr{C}(Y)$ is l.s.c., then F has a continuous selection.*

While lower semicontinuity of F is *sufficient* for the existence of a continuous selection, it is in general not necessary. Deutsch and Kenderov have characterized almost lower semicontinuity and in the process showed that it is a *necessary* condition for the existence of a continuous selection.

1.2. Theorem (Deutsch and Kenderov [5]). *Let $F\colon X \to \mathscr{C}(Y)$. Then F is a.l.s.c. if and only if for each $\varepsilon > 0$, F has a continuous ε-approximate selection.*

In particular, a.l.s.c. is weaker than l.s.c. It was also observed in [5] that, in certain cases, the almost lower semicontinuity of F is *equivalent* to the existence of a continuous selection for F. (This is the case, for example, when Y is one dimensional and $F\colon X \to \mathscr{C}(Y)$ has bounded images, or when the set of points in X where $F(x)$ is a singleton is dense in X.) This has some interesting ramifications when F is a metric projection.

Recall that the *metric projection* or nearest-point mapping onto a (finite-dimensional) subspace Y of the normed linear space X is the mapping $P_Y\colon X \to 2^Y$ defined by

$$P_Y(x) := \{y \in Y \mid \|x - y\| = d(x, Y)\}.$$

That is, $P_Y(x)$ is the set of all best approximations (= nearest points) in Y to x. It is well known that P_Y has nonempty, closed, convex, and bounded images so $P_Y\colon X \to \mathscr{C}(Y)$.

The main results of the paper can now be briefly summarized. Theorem 2.4 characterizes when F has a continuous selection and is a kind of "dual" to a theorem of Michael (see Theorem 2.6) where lower semicontinuity was characterized. A characterization of almost lower semicontinuity by a property which is (formally) stronger than that of Theorem 1.2 is given in Theorem 3.3. Theorem 4.5 is a geometric characterization of those one-dimensional subspaces Y of a normed linear space X such that P_Y admits a continuous selection. Using Theorem 4.5, we give in Theorem 5.1 (resp. Theorem 6.3) an *intrinsic* characterization of those one-dimensional subspaces Y in the classical Banach space $C_0(T)$, T locally compact (resp. $L_1(T, \mathscr{S}, \mu)$) such that P_Y admits a continuous selection. The former result generalizes a theorem of Lazar, Morris, and Wulbert [10], who proved the particular case when T is compact by a more involved argument. The latter result generalizes a theorem of Lazar [9], who proved the special case of l_1-space. In contrast to our unified approach, the methods used in [10] are completely different from those used in [9].

2. Lower Semicontinuity and Continuous Selections

In this section we study some connections between lower semicontinuity and certain kinds of continuous selections. For ease of reference, we collect some basic known facts about constructing new l.s.c. maps from given ones in the next proposition (see, e.g., [11]).

2.1. Proposition. (1) *If $F: X \to 2^Y$ is l.s.c., so are the mappings $\bar{F}$,* $\mathrm{co}(F)$, *and* $\overline{\mathrm{co}}(F)$. (*Here $\bar{F}(x) := \overline{F(x)}$, where the bar denotes closure and* $\mathrm{co}(F)(x) := \mathrm{co}(F(x))$, *where* "co" *denotes "convex hull of."*)

(2) *If $G_i: X \to 2^Y$ is l.s.c. for each $i \in I$, then $\bigcup_{i \in I} G_i$ is also l.s.c., where $(\bigcup_{i \in I} G_i)(x) := \bigcup_{i \in I} G_i(x)$.*

(3) *Let $F: X \to 2^Y$ be l.s.c., A a nonempty closed set in X, and $\alpha: A \to Y$ a continuous function with $\alpha(x) \in F(x)$ for every $x \in A$. Then the mapping G defined on X by*

$$G(x) = \begin{cases} F(x) & \text{if } x \in X \backslash A \\ \{\alpha(x)\} & \text{if } x \in A \end{cases}$$

is l.s.c.

2.2. Definition. Let F, $G: X \to 2^Y$. G is called a *submap* of F, denoted $G \subset F$, provided $G(x) \subset F(x)$ for every $x \in X$.

If $\mathscr{S}$ is a collection of submaps of F, a mapping $G_0 \in \mathscr{S}$ is called a *maximal submap* for $\mathscr{S}$ if $G \in \mathscr{S}$ and $G_0 \subset G$ implies $G = G_0$.

2.3. LEMMA. *If $F: X \to \mathscr{C}(Y)$ has a maximal l.s.c. submap, it is unique.*

Proof. Let G_1 and G_2 both be maximal l.s.c. submaps of F. Then the map $G = G_1 \cup G_2$ is l.s.c. by 2.1(2) and the map $\overline{\text{co}}(G)$ is l.s.c. by 2.1(1). But $G_i \subset \overline{\text{co}}(G) \subset F$ for $i = 1, 2$. Since G_i is maximal for $i = 1, 2$, it follows that $\overline{\text{co}}(G) = G_i$ for $i = 1, 2$ and hence $G_1 = G_2$. ∎

The next result shows that while l.s.c. is not necessary to guarantee a continuous selection, some *submap* must be l.s.c.!

2.4. THEOREM. *Let Y be complete and F: $X \to \mathscr{C}(Y)$. Then the following statements are equivalent.*

(1) *F has a continuous selection*;

(2) *F has "local" continuous selections, i.e., for each $x_0 \in X$, there is a neighborhod $U(x_0)$ of x_0 and a continuous function f_{x_0}: $U(x_0) \to Y$ such that $f_{x_0}(x) \in F(x)$ for every $x \in U(x_0)$*

(3) *F gas a (unique) maximal l.s.c. submap*

(4) *F has a l.s.c. submap.*

Proof. (1) ⇒ (2). This is obvious since the restriction of a continuous selection f to any neighborhood $U(x_0)$ of x_0 can be chosen for f_{x_0}.

(2) ⇒ (1). Suppose (2) holds. Since X is paracompact, the open cover $\{U(x_0) \mid x_0 \in X\}$ of X has a locally finite refinement $\{V_i \mid i \in I\}$. For each $i \in I$, choose $x_i \in X$ so that $V_i \subset U(x_i)$. Using paracompactness, we can choose a partition of unity $\{p_i \mid i \in I\}$ subordinate to $\{V_i \mid i \in I\}$. That is, each function p_i: $X \to [0, 1]$ is continuous, $p_i = 0$ off V_i, and $\sum_{i \in I} p_i(x) = 1$ for all $x \in X$. Define f on X by

$$f(x) = \sum_{i \in I} p_i(x) f_{x_i}(x), \qquad x \in X,$$

Given any $x \in X$, there is a neighborhood of x which intersects only finitely many of the V_i so $x \in V_i$ for only a finite set of indices $I(x)$ in I. Thus f is well-defined, continuous, and has range in Y. Further,

$$f(x) = \sum_{i \in I(x)} p_i(x) f_{x_i}(x) \in \text{co}(F(x)) = F(x)$$

so f is a continuous selection for F.

(1) ⇒ (3). Assume F has a continuous selection f and let $\mathscr{L}$ denote the collection of all l.s.c. submaps of F. $\mathscr{L} \neq \varnothing$ since $f \in \mathscr{L}$. Define

$G_0 = \bigcup \{G \mid G \in \mathscr{L}\}$. By Proposition 2.1(2), G_0 is l.s.c. Further, the map $H = \overline{\text{co}}(G_0)$ is l.s.c. by 2.1(1). Also, $G \subset H$ for every $G \in \mathscr{L}$. That is, H is a maximal l.s.c. submap for $\mathscr{L}$. By Lemma 2.3, H is unique.

(3) $\Rightarrow$ (4). Obvious.

(4) $\Rightarrow$ (1). Suppose G is a l.s.c. submap of F. Then $G_0 := \overline{\text{co}}(G)$ is a l.s.c. submap also. By Michael's selection theorem 1.1, G_0 has a continuous selection which is obviously also a continuous selection for F. ▌

One can give an explicit formula for the maximal l.s.c. submap.

2.5. Proposition. *Suppose $F: X \to \mathscr{C}(Y)$ has a continuous selection. Define a mapping F^* on X by*

$$F^*(x) := \{ y \in F(x) \mid \text{there is a continuous selection } f \text{ for } F \text{ with } f(x) = y \}. \tag{2.5.1}$$

That is,

$$F^* = \bigcup \{ f \mid f \text{ is a continuous selection for } F \}.$$

Then F^* is the unique maximal l.s.c. submap of F. In particular, $F^*: X \to \mathscr{C}(Y)$.

Proof. Clearly F^* is l.s.c. by Proposition 2.1(2), and F^* is a submap of F. If F^* were not maximal, there would exist a l.s.c. submap F_0 of F with $F^* \subset F_0$ and $F^* \neq F_0$. By replacing F_0 with $\overline{\text{co}}(F_0)$, we may assume that F_0 has closed convex images. Then there is an $x_0 \in X$ with $F_0(x_0) \backslash F^*(x_0) \neq \varnothing$. Choose $y_0 \in F_0(x_0) \backslash F^*(x_0)$ and define H on X by

$$H(x) = \begin{cases} F_0(x) & \text{if } x \neq x_0 \\ \{y_0\} & \text{if } x = x_0. \end{cases}$$

By 2.1(3), $H: X \to \mathscr{C}(Y)$ is l.s.c. so, by Theorem 1.1, H has a continuous selection $f: X \to Y$. Since H is a submap of F, f is also a continuous selection for F with $f(x_0) = y_0$. Hence $y_0 \in F^*(x_0)$, a contradiction. Thus F^* is maximal and the proof is complete. ▌

Theorem 2.4 equates the existence of a continuous selection for F with the existence of a lower semicontinuous submap of F. There is an interesting "dual" result to this which equates the lower semicontinuity of F with the existence of extensions of continuous selections for certain restrictions of F. It is essentially due to Michael and can be stated as follows.

2.6. THEOREM (Michael [11]). *Let Y be complete and $F: X \to \mathscr{C}(Y)$. Then the following statements are equivalent.*

(1) *F is l.s.c.;*

(2) *For each closed subset A of X, each continuous selection for $F|_A$ has an extension to a continuous selection for F;*

(3) *For each $x_0 \in X$ and $y_0 \in F(x_0)$, there exists a continuous selection f for F with $f(x_0) = y_0$, i.e., $F = F^*$.*

[Actually, Michael verified the implications (1)$\Leftrightarrow$(2) and (3)$\Rightarrow$(1) in [11]. To verify (2)$\Rightarrow$(3), let $A = \{x_0\}$, $f(x_0) = y_0$, and observe that f is (trivially) a continuous selection for $F|_A$. Thus it has an extension to a continuous selection for F.]

It follows that if F is l.s.c. and there is some point x_0 where $F(x_0)$ is not a singleton, the F can *not* have a unique continuous selection. More precisely, we have the next corollary.

2.7. COROLLARY. *Assume Y is complete, $F: X \to \mathscr{C}(Y)$ is l.s.c., and $F(x_0)$ is not a singleton for some $x_0 \in X$. Then F has at least* card$(F(x_0))$ *distinct continuous selections.*

Proof. By Theorem 2.6, for each $y_0 \in F(x_0)$, F has a continuous selection f with $f(x_0) = y_0$. There are obviously at least card$(F(x_0))$ such selections. ∎

For certain mappings (which include metric projections onto subspaces), it is possible to characterize lower semicontinuity in terms of the existence of a continuous selection having an additional property.

2.8. THEOREM. *Let Y be a complete subspace of the normed linear space X and $F: X \to \mathscr{C}(Y)$. Assume that*

$$\ker F := \{x \in X \mid 0 \in F(x)\}$$

is closed and F is "additive modulo Y," i.e., for each $x \in X$ and $y \in Y$,

$$F(x + y) = F(x) + y. \tag{2.8.1}$$

Then F is l.s.c. if and only if F has a continuous selection f which is "kernel-preserving," i.e., $f(x) = 0$ for every $x \in \ker F$.

Proof. Suppose F is l.s.c. Then the restriction mapping $F|_{\ker F}$ has a continuous selection (viz. $g = 0$). By Theorem 2.6, F has a continuous selection f which is an extension of g. Thus f is kernel-preserving.

Conversely, suppose F has a continuous selection which is kernel-preserving. Let $x_0 \in X$ and let W be an open set in Y with $F(x_0) \cap W \neq \emptyset$. We will show that there is a neighborhood U of x_0 such that $F(x) \cap W \neq \emptyset$ for all $x \in U$. Let $y_0 \in F(x_0) \cap W$. Then there is an $\varepsilon > 0$ such that $B_\varepsilon(y_0) \subset W$. Now $0 \in F(x_0) - y_0 = F(x_0 - y_0)$ so $f(x_0 - y_0) = 0$. Choose a neighborhood U_0 of $z_0 := x_0 - y_0$ so that

$$f(z) \in B_\varepsilon(f(z_0)) = B_\varepsilon(0)$$

for every $z \in U_0$. Then $U := U_0 + y_0$ is a neighborhood of x_0 and for each $x \in U$, $x = z + y_0$ for some $z \in U_0$. Thus

$$\begin{aligned} f(z) + y_0 &\in [F(z) + y_0] \cap [B_\varepsilon(0) + y_0] \\ &= F(z + y_0) \cap B_\varepsilon(y_0) = F(x) \cap B_\varepsilon(y_0) \subset F(x) \cap W \end{aligned}$$

implies that $F(x) \cap W \neq \emptyset$ for every $x \in U$. This proves that F is l.s.c. at x_0. Since x_0 was arbitrary, F is l.s.c. ▌

Recall that a subspace Y of the normed linear space X is called *proximinal* if

$$P_Y(x) := \{y \in Y \mid \|x - y\| = d(x, Y)\}$$

is nonempty for every $x \in X$. For example, any finite-dimensional subspace is proximinal. It is well known and easy to prove that P_Y is additive modulo Y and

$$\ker P_Y := \{x \in X \mid 0 \in P_Y(x)\} = \{x \in X \mid \|x\| = d(x, Y)\}$$

is closed. Thus, as consequence of Theorem 2.8, we immediately obtain the following result of Krüger.

2.9. THEOREM (Krüger [8]). *Let Y be a complete proximinal subspace of the normed linear space X. Then P_Y is l.s.c. if and only if P_Y has a continuous selection f with $f(x) = 0$ for every $x \in \ker P_Y$.*

Remarks. (1) The proof of Theorem 2.8 is an obvious generalization of Krüger's original proof for the special case of metric projections. (His proof also used Proposition 2.1(3).)

(2) There are mappings which are not metric projections but which satisfy the hypothesis of Theorem 2.8. For example, $F(x) := P_Y(x) + f(x)$, where $f: X \to Y$ is any continuous function satisfying $f(x + y) = f(x)$ for every $x \in X$ and $y \in Y$.

3. Almost Lower Semicontinuity and Continuous Selections

The distinction between lower semicontinuity and almost lower semicontinuity is further elucidated in the next lemma.

3.1. Lemma. *Let* $F\colon X \to 2^Y$, $x_0 \in X$, *and consider the following statements.*

(1) *F is l.s.c. at x_0:*

(2) *For each $y_0 \in F(x_0)$ and $\varepsilon > 0$, there exists a neighborhood U of x_0 such that*

$$y_0 \in \bigcap_{x \in U} B_\varepsilon(F(x)); \tag{3.1.1}$$

(3) *For each $y_0 \in F(x_0)$ and each net $x_n \to x_0$,*

$$d(y_0, F(x_n)) \to 0; \tag{3.1.2}$$

(4) *There exists $y_0 \in F(x_0)$ such that for each net $x_n \to x_0$, $d(y_0, F(x_n)) \to 0$;*

(5) *There exists $y_0 \in F(x_0)$ such that for each $\varepsilon > 0$, there exists a neighborhood U of x_0 with*

$$y_0 \in \bigcap_{x \in U} B_\varepsilon(F(x));$$

(6) *For each $\varepsilon > 0$, there exists a neighborhood U of x_0 such that*

$$F(x_0) \cap \left[\bigcap_{x \in U} B_\varepsilon(F(x))\right] \neq \varnothing; \tag{3.1.3}$$

(7) *F is a.l.s.c. at x_0.*

Then

$$(1) \Leftrightarrow (2) \Leftrightarrow (3) \Rightarrow (4) \Leftrightarrow (5) \Rightarrow (6) \Leftrightarrow (7).$$

Moreover, if $F(x_0)$ is compact, then $(6) \Rightarrow (5)$ and the last four statements are equivalent. If $F(x_0)$ is a singleton, then all seven statements are equivalent. If X is a metric space, then "net" may be replaced by "sequence" in (3) and (4).

Proof. The equivalence of the first three statements is well known and goes back at least to Hahn [7].

The implication (3) $\Rightarrow$ (4) is obvious.

(4) $\Rightarrow$ (5). Suppose (5) fails. Then for each $y_0 \in F(x_0)$ there exists an $\varepsilon > 0$ such that for each neighborhood U of x_0, there is an $x_U \in U$ with $y_0 \notin B_\varepsilon(F(x_U))$, i.e., $d(y_0, F(x_U)) \geqslant \varepsilon$. Then the net $\{x_U\}$ converges to x_0 but (3.1.2) fails. Thus (4) fails.

(5) $\Rightarrow$ (4). Assume (5) holds and let $\{x_n\}$ be a net converging to x_0. For any $\varepsilon > 0$, choose a neighborhood U of x_0 so that (3.1.1) holds. Then $x_n \in U$ eventually so $y_0 \in B_\varepsilon(F(x_n))$ eventually, i.e., $d(y_0, F(x_n)) < \varepsilon$ eventually. Thus (3.1.2) holds.

The implications (5) $\Rightarrow$ (6) and (6) $\Rightarrow$ (7) are obvious.

(7) $\Rightarrow$ (6). Suppose F is a.l.s.c. at x_0 and let $\varepsilon > 0$. Then there is a neighborhood U of x_0 such that

$$\bigcap_{x \in U} B_{\varepsilon/2}(F(x)) \neq \varnothing.$$

Select any $y \in \bigcap_{x \in U} B_{\varepsilon/2}(F(x))$. For each $x \in U$, choose $y_x \in F(x)$ such that $\|y - y_x\| < \varepsilon/2$. It follows that

$$\|y_{x_0} - y_x\| \leqslant \|y_{x_0} - y\| + \|y - y_x\| < \varepsilon$$

for each $x \in U$. Hence $y_{x_0} \in \bigcap_{x \in U} B_\varepsilon(F(x))$ and so (3.1.3) holds. This verifies (6).

Now assume $F(x_0)$ is compact and (6) holds. We will show that (5) holds. For each integer n, there is a neighborhood U_n of x_0 such that

$$A_n := F(x_0) \cap \left[\bigcap_{x \in U_n} B_{1/n}(F(x)) \right] \neq \varnothing.$$

Select any $y_n \in A_n$. By compactness of $F(x_0)$, the sequence $\{y_n\}$ has a cluster point $y_0 \in F(x_0)$. Now let $\{x_\alpha\}$ be a net in X with $x_\alpha \to x_0$. Then for each n, $x_\alpha \in U_n$ eventually. Hence $y_n \in B_{1/n}(F(x_\alpha))$ for α eventually so

$$\begin{aligned} d(y_0, F(x_\alpha)) &\leqslant d(y_0, y_n) + d(y_n, F(x_\alpha)) \\ &\leqslant d(y_0, y_n) + 1/n. \end{aligned}$$

Since y_0 is a cluster point of $\{y_n\}$, for each $\varepsilon > 0$ and integer n with $n > 2/\varepsilon$, there is an $n_0 \geqslant n$ such that $d(y_0, y_{n_0}) < \varepsilon/2$. Hence $d(y_0, F(x_\alpha)) < \varepsilon$ eventually. This verifies (5).

The remainder of the proof is easy. ∎

An example given at the end of [5] shows that the implication (4) $\Rightarrow$ (3) is false, even when $F(x_0)$ is compact and F is a metric projection onto a one-dimensional subspace. Also, P. Kenderov has communicated an example to us showing that the implication (6) ⇒ (5) is false when $F(x_0)$ is not compact.

3.2. DEFINITION. A subset I of X is called *discrete* if $I \backslash \{x\}$ is closed for each $x \in I$. Equivalently, I is discrete iff I has no accumulation points iff I is closed and, for each $x \in I$, there exists a neighborhood U of x such that $U \cap I = \{x\}$.

Note that each discrete set is necessarily closed, and every finite set is discrete. The next result is a strengthening of Theorem 1.2.

3.3. THEOREM. *Let $F: X \to \mathscr{C}(Y)$. Then the following statements are equivalent.*

(1) *F is a.l.s.c.;*

(2) *For each $x_0 \in X$ and $\varepsilon > 0$, there is a continuous ε-approximate selection f for F with $f(x_0) \in F(x_0)$;*

(3) *For each discrete set I in X and $\varepsilon > 0$, there exists a continuous ε-approximate selection f for F with $f(x) \in F(x)$ for all $x \in I$.*

Proof. (1) ⇒ (2). Assume F is a.l.s.c. and fix any $x_0 \in X$. By Theorem 1.2, for every $\varepsilon > 0$, there exists a continuous $\varepsilon/2$-approximate selection f_1 for F. Choose any $y_0 \in F(x_0)$ with $f_1(x_0) \in B_{\varepsilon/2}(y_0)$. Define f on X by

$$f(x) = f_1(x) + y_0 - f_1(x_0).$$

Then f is continuous,

$$f(x) \in B_{\varepsilon/2}(F(x)) + B_{\varepsilon/2}(0) \subset B_\varepsilon(F(x))$$

for every $x \in X$, and $f(x_0) = y_0 \in F(x_0)$. Thus f is a continuous ε-approximate selection for F with $f(x_0) \in F(x_0)$. This proves (2).

(2) ⇒ (3). Assume (2) holds, let I be a discrete set in X, and $\varepsilon > 0$. Then for each $x \in I$, there exists a continuous ε-approximate selection f_x for F such that $f_x(x) \in F(x)$. Define

$$U(x) := X \backslash (I \backslash \{x\}) = (X \backslash I) \cup \{x\}.$$

Then $\{U(x) \mid x \in I\}$ is an open cover of X. Let $\{V_j \mid j \in J\}$ be a locally finite refinement of $\{U(x) \mid x \in I\}$. For each $j \in J$, choose $x_j \in I$ so that $V_j \subset U(x_j)$. Choose a partition of unity $\{p_j \mid j \in J\}$ subordinate to the cover $\{V_j \mid j \in J\}$. Then each p_j is continuous, $0 \leqslant p_j \leqslant 1$, $p_j = 0$ off V_j, and $\sum_{j \in J} p_j(x) = 1$ for every $x \in X$. In particular, $p_j(x)=0$ for all $x \notin U(x_j)$ so $p_j(x)=0$ for all $x \in I \backslash \{x_j\}$ and hence $p_j(x_j)=1$. Define f on X by

$$f(x)=\sum_{j \in J} p_j(x) f_{x_j}(x), \qquad x \in X.$$

Then f is continuous,

$$f(x) \in \operatorname{co}\{f_{x_j}(x) \mid j \in J\} \subset \operatorname{co}(B_\varepsilon(F(x)))=B_\varepsilon(F(x))$$

for all $x \in X$, and, if $x \in I$, $f(x)=f_x(x) \in F(x)$. Thus f is a continuous ε-approximate selection for F such that $f(x) \in F(x)$ for every $x \in I$. This proves (3).

The implication (3) ⇒ (1) follows by Theorem 1.2. ∎

4. Continuous Selections for Metric Projections onto One-Dimensional Subspaces

In this section we will be exclusively connerned with the metric projection onto a one-dimensional subspace. A geometric characterization of when such a metric projection admits a continuous selection is obtained. In the last two sections, this result will be used to obtain intrinsic characterizations of the one-dimensional subspaces in $C_0(T)$ and $L_1(\mu)$ whose metric projections admit continuous selections.

Let X be a normed linear space, $y_1 \in X \backslash \{0\}$, and let $[y_1]$ denote the one-dimensional subspace spanned by y_1:

$$[y_1]:=\{\alpha y_1 \mid \alpha \in \mathbf{R}\}.$$

The following result of [5] is central to our development. It essentially states that a rather weak continuity property of $P_{[y_1]}$ is equivalent to the existence of a continuous selection.

4.1. Theorem [5]. *The following statements are equivalent.*

(1) $P_{[y_1]}$ *has a continuous selection*;

(2) $P_{[y_1]}$ *is a.l.s.c.*;

(3) $P_{[y_1]}$ *is 2-l.s.c.*

Recall [5] that $P_{[y_1]}$ is 2-*l.s.c.* at $x_0 \in X$ if, for each $\varepsilon>0$, there exists a neighborhood U of x_0 such that

$$B_\varepsilon(P_{[y_1]}(x_1)) \cap B_\varepsilon(P_{[y_1]}(x_2)) \neq \varnothing \tag{4.1.1}$$

whenever $x_1, x_2 \in U$.

If I is any (bounded or unbounded) closed interval in $\mathbf{R}$, we define

$$Iy_1 := \{\alpha y_1 \mid \alpha \in I\}.$$

In particular, $\mathbf{R}y_1 = (-\infty, \infty)\, y_1$ is what we earlier denoted by $[y_1]$. It is well known that for every $x \in X$,

$$P_{[y_1]}(x) = I_x\, y_1$$

for some compact interval I_x depending on x.

Fix any $x \in X$. It is not hard, using the convexity of the function $f(\alpha) = \|x + \alpha y_1\|$, $\alpha \in \mathbf{R}$, to verify that

$$P_{[y_1]}(x) \supset [-1, 1]\, y_1 \tag{4.1.2}$$

holds if and only if

$$\|x\| = \|x + y_1\| = \|x - y_1\|. \tag{4.1.3}$$

It will be convenient to single out those $x \in X$ which have either of the equivalent properties (4.1.2) or (4.1.3). Define

$$\begin{aligned} P^0_{[y_1]} &:= \{x \in X \mid P_{[y_1]}(x) \supset [-1, 1]\, y_1\} \\ &= \{x \in X \mid \|x\| = \|x + y_1\| = \|x - y_1\|\}. \end{aligned} \tag{4.1.4}$$

4.2. Lemma. *The following statements are equivalent.*

(1) $P_{[y_1]}$ *is 2-l.s.c.*;

(2) $P_{[y_1]}$ *is 2-l.s.c. on* $\ker P_{[y_1]}$;

(3) $P_{[y_1]}$ *is 2-l.s.c. on* $P^0_{[y_1]}$.

Proof. Since $P^0_{[y_1]} \subset \text{ket}\, P_{[y_1]}$, the implications (1) $\Rightarrow$ (2) $\Rightarrow$ (3) are obvious.

(3) $\Rightarrow$ (1). Suppose $P = P_{[y_1]}$ fails to be 2-l.s.c. at some $x_0 \in X$. Then there exist $\varepsilon > 0$ and sequences $x_n \to x_0$ and $x'_n \to x_0$ with

$$B_\varepsilon(P(x_n)) \cap B_\varepsilon(P(x'_n)) = \varnothing$$

for all n. Now $P(x_0) = [\alpha_0, \beta_0]\, y_1$ for some $\beta_0 > \alpha_0$. Set $y_0 = \frac{1}{2}(\alpha_0 + \beta_0)\, y_1$ and $\delta = \frac{1}{2}(\beta_0 - \alpha_0)$. If $\delta \geqslant 1$, let $z_0 = x_0 - y_0$, $z_n = x_n - y_0$, and $z'_n = x'_n - y_0$. Then $z_n \to z_0$, $z'_n \to z_0$, and

$$\begin{aligned} B_\varepsilon(P(z_n)) \cap B_\varepsilon(P(z'_n)) &= B_\varepsilon(P(x_n) - y_0) \cap B_\varepsilon(P(x'_n) - y_0) \\ &= B_\varepsilon(P(x_n)) \cap B_\varepsilon(P(x'_n)) - y_0 = \varnothing \end{aligned}$$

for all n. Also,

$$P(z_0) = P(x_0) - y_0 = [\alpha_0, \beta_0]\, y_1 - \tfrac{1}{2}(\alpha_0 + \beta_0)\, y_1$$
$$= [-\delta, \delta]\, y_1 \supset [-1, 1]\, y_1.$$

If $0 < \delta < 1$, let $z_0 = (1/\delta)(x_0 - y_0)$, $z_n = (1/\delta)(x_n - y_0)$, and $z'_n = (1/\delta)(x'_n - y_0)$. Then $z_n \to z_0$, $z'_n \to z_0$, and

$$\begin{aligned} B_\varepsilon(P(z_n)) \cap B_\varepsilon(P(z'_n)) &= B_\varepsilon\left[\frac{1}{\delta} P(x_n) - \frac{1}{\delta} y_0\right] \cap B_\varepsilon\left[\frac{1}{\delta} P(x'_n) - \frac{1}{\delta} y_0\right] \\ &= \frac{1}{\delta}\left[B_{\delta\varepsilon}(P(x_n)) \cap B_{\delta\varepsilon}(P(x'_n))\right] - \frac{1}{\delta} y_0 \\ &\subset \frac{1}{\delta}\left[B_\varepsilon(P(x_n)) \cap B_\varepsilon(P(x'_n))\right] - \frac{1}{\delta} y_0 \\ &= \varnothing \end{aligned}$$

for all n. Also,

$$\begin{aligned} P(z_0) &= \frac{1}{\delta} P(x_0) - \frac{1}{\delta} y_0 = \frac{1}{\delta}[\alpha_0, \beta_0]\, y_1 - \frac{1}{2\delta}(\alpha_0 + \beta_0)\, y_1 \\ &= \frac{1}{\delta}[-\delta, \delta]\, y_1 = [-1, 1]\, y_1. \end{aligned}$$

In either case, we see that P is not 2-l.s.c. at z_0 and $z_0 \in P^0_{[y_1]}$. Thus (3) fails. ∎

For a given $x \in X$, the set of *peaking functionals* for x is the subset of the dual space X^* defined by

$$J(x) := \{x^* \in X_1^* \mid x^*(x) = \|x\|\},$$

where $X_1^* := \{x^* \in X^* \mid \|x^*\| \leqslant 1\}$. The annihilator of the subspace $[y_1]$ is the set

$$y_1^\perp := \{x^* \in X^* \mid x^*(y_1) = 0\}.$$

The extreme points of any set B in the dual space X^* will be denoted by ext B. Since $J(x)$ is a nonempty weak* compact extremal subset of X_1^* for any x, it follows that $J(x)$ has extreme points and ext $J(x) \subset$ ext X_1^*.

4.3. LEMMA. *Let* $P_{[y_1]}(x_0)=[\alpha_0, \beta_0]\, y_1$ *with* $\alpha_0 \leqslant \beta_0$ *and* $\alpha \in \mathbf{R}$. *Then*:

(1) $\alpha<\alpha_0$ *if and only if*

$$J(x_0-\alpha y_1) \subset \{x^* \in X^* \mid x^*(y_1)>0\}. \tag{4.3.1}$$

(2) $\alpha>\beta_0$ *if and only if*

$$J(x_0-\alpha y_1) \subset \{x^* \in X^* \mid x^*(y_1)<0\}. \tag{4.3.2}$$

(3) *If* $\alpha \in (\alpha_0, \beta_0)$, *then*

$$J(x_0-\alpha y_1) \subset \{x^* \in X^* \mid x^*(y_1)=0\}. \tag{4.3.3}$$

(4) *If* (4.3.3) *holds, then*

$$\alpha \in [\alpha_0. \beta_0]. \tag{4.3.4}$$

(5) *If* $\alpha=\alpha_0$ *(resp.* $\alpha=\beta_0$*) and* $\alpha_0 \neq \beta_0$, *then*

$$J(x_0-\alpha y_1) \subset \{x^* \in X^* \mid x^*(y_1) \geqslant 0\} \tag{4.3.5}$$

(resp. $J(x_0-\alpha y_1) \subset \{x^* \in X^* \mid x^*(y_1) \leqslant 0\}$*).*

Proof. Let $\alpha \in \mathbf{R}$. Then for each $\gamma \in [\alpha_0, \beta_0]$ and $x^* \in J(x_0-\alpha y_1)$, we have

$$\begin{aligned}\|x_0-\alpha y_1\| &= x^*(x_0-\alpha y_1)=x^*(x_0-\gamma y_1)+(\gamma-\alpha)\, x^*(y_1)\\ &\leqslant \|x_0-\gamma y_1\|+(\gamma-\alpha)\, x^*(y_1)\\ &= d(x_0, [y_1])+(\gamma-\alpha)\, x^*(y_1).\end{aligned}$$

Hence setting

$$\delta(\alpha):=\|x_0-\alpha y_1\|-d(x_0[y_1]),$$

we have that

$$\delta(\alpha) \leqslant (\gamma-\alpha)\, x^*(y_1) \tag{4.3.6}$$

for every $\gamma \in [\alpha_0, \beta_0]$ and $x^* \in J(x_0-\alpha y_1)$.

(1) Assume $\alpha<\alpha_0$. Then $\delta(\alpha)>0$ and (4.3.6) implies that $(\gamma-\alpha)\, x^*(y_1)>0$ for all $\gamma \in [\alpha_0, \beta_0]$ and $x^* \in J(x_0-\alpha y_1)$. Hence $x^*(y_1)>0$ for all $x^* \in J(x_0-\alpha y_1)$ and (4.3.1) holds.

Conversely, suppose (4.3.1) holds. We will show that $\alpha<\alpha_0$. From (4.3.6) we deduce that $\gamma-\alpha \geqslant 0$ for all $\gamma \in [\alpha_0, \beta_0]$. In particular, $\alpha \leqslant \alpha_0$. If

$\alpha = \alpha_0$, then by the Hahn–Banach theorem, there exists $x^* \in J(x_0 - \alpha y_1) \cap y_1^\perp$. But this contradicts (4.3.1). Thus $\alpha < \alpha_0$.

(2) The proof is similar to (1).

(3) Let $\alpha \in (\alpha_0, \beta_0)$. Then by (4.3.6), we see that $(\gamma - \alpha)\, x^*(y_1) \geqslant 0$ for all $x^* \in J(x_0 - \alpha y_1)$ and $\gamma \in [\alpha_0, \beta_0]$. It follows that $x^*(y_1) = 0$ for all $x^* \in J(x_0 - \alpha y_1)$. That is, (4.3.3) holds.

(4) Suppose (4.3.3) holds. From (1) and (2) it follows that $\alpha_0 \leqslant \alpha \leqslant \beta_0$. That is, (4.3.4) holds.

(5) Let $\alpha = \alpha_0$ and $\alpha_0 \neq \beta_0$. By (4.3.6), we see that $(\gamma - \alpha_0)\, x^*(y_1) \geqslant 0$ for all $\gamma \in [\alpha_0, \beta_0]$ and all $x^* \in J(x_0 - \alpha_0 y_1)$. In particular, $(\beta_0 - \alpha_0)\, x^*(y_1) \geqslant 0$ or $x^*(y_1) \geqslant 0$ for all $x^* \in J(x_0 - \alpha_0 y_1)$. This verifies (4.3.6). The case $\alpha = \beta_0$ is similar. ∎

4.4. COROLLARY. *Let* $P(x_0) = [\alpha_0, \beta_0]\, y_1$, *with* $\alpha_0 \leqslant \beta_0$.

(1) *If*

$$J(x_0 - \alpha y_1) \cap \{x^* \in X^* \mid x^*(y_1) > 0\} \neq \varnothing, \tag{4.4.1}$$

then $\alpha \leqslant \alpha_0$.

(2) *If*

$$J(x_0 - \alpha y_1) \cap \{x^* \in X^* \mid x^*(y_1) < 0\} \neq \varnothing, \tag{4.4.2}$$

then $\alpha \geqslant \beta_0$.

Proof. (1) If (4.4.1) holds, then by Lemma 4.3(2) and (3), we see that $\alpha \leqslant \beta_0$ and $\alpha \notin (\alpha_0, \beta_0)$. Thus $\alpha \leqslant \alpha_0$ or $\alpha = \beta_0$. If $\alpha_0 = \beta_0$, we're done. If $\alpha_0 \neq \beta_0$ and $\alpha = \beta_0$, then by 4.3(5),

$$J(x_0 - \alpha y_1) \subset \{x^* \in X^* \mid x^*(y_1) \leqslant 0\},$$

which contradicts (4.4.1). Thus, in every case, $\alpha \leqslant \alpha_0$.

(2) The proof is similar. ∎

In the product space $X \times X^*$, we assume that X has its norm topology and X^* its weak* topology. Thus a net $\{(x_n, x_n^*)\}$ in $X \times X^*$ converges to (x_0, x_0^*), denoted $(x_n, x_n^*) \to (x_0, x_0^*)$, if and only if $\|x_n - x_0\| \to 0$ and $x_n^* \to x_0^*$ weak*, i.e., $x_n^*(x) \to x_0^*(x)$ for every $x \in X$.

For any $x_0 \in \ker P_{[y_1]}$ and $\sigma \in \{+1, -1\}$, we define a subset of X^* as follows.

$$\begin{aligned} A(x_0, \sigma) := y_1^\perp \cap J(x_0) \cap \{x_0^* \in X_1^* \mid & \text{ there is a net } \{(x_n, x_n^*)\} \\ & \text{in } X \times X^* \text{ with } (x_n, x_n^*) \to (x_0, x_0^*),\ x_n^* \in \operatorname{ext} J(x_n + \sigma y_1), \\ & \text{and } x_n^*(\sigma y_1) < 0 \text{ for all } n\}. \end{aligned}$$

We now state the main result of this section.

4.5. THEOREM. *The following statements are equivalent.*

(1) $P_{[y_1]}$ *does not have a continuous selection;*

(2) *There exist* $x_0 \in \ker P_{[y_1]}$, *disjoint compact intervals* I *and* I' *in* $\mathbf{R}$, *and sequences* $\{x_n\}$ *and* $\{x_n'\}$ *converging to* x_0 *such that for every* n,

$$P_{[y_1]}(x_n) \subset I y_1, \qquad P_{[y_1]}(x_n') \subset I' y_1; \tag{4.5.1}$$

(3) *There exists* $x_0 \in X$ *with* $\|x_0\| = \|x_0 - y_1\| = \|x_0 + y_1\|$ *such that* $A(x_0, +1) \neq \varnothing$ *and* $A(x_0, -1) \neq \varnothing$.

Proof. Let $P = P_{[y_1]}$.

(1) $\Rightarrow$ (2). If P does not have a continuous selection, then (by Lemmas 4.1 and 4.2) P fails to be 2-l.s.c. at some $x_0 \in \ker P$ with $P(x_0) = [\alpha_0, \beta_0]\, y_1$ and $\alpha_0 < \beta_0$. Thus there exist sequences $\{x_n\}$ and $\{x_n'\}$ both converging to x_0 such that

$$B_\varepsilon(P(x_n)) \cap B_\varepsilon(P(x_n')) = \varnothing \qquad \text{for all } n.$$

Letting $P(x_n) = I_n y_1$ and $P(x_n') = I_n' y_1$ for some compact intervals I_n and I_n' in $\mathbf{R}$, it follows that

$$B_\varepsilon(I_n y_1) \cap B_\varepsilon(I_n' y_1) = \varnothing \qquad \text{for all } n. \tag{4.5.2}$$

We may assume $\sup I_n < \inf I_n'$ for all n; in fact, by (4.5.2) we have (assuming, as we may, that $\|y_1\| = 1$) that

$$\sup I_n + 2\varepsilon < \inf I_n'. \tag{4.5.3}$$

Since these intervals are uniformly bounded, by passing to a subsequence, we deduce that there exist intervals J and J' such that $I_n \subset B_{\varepsilon/2}(J)$, $I_n' \subset B_{\varepsilon/2}(J')$, $J \subset B_{\varepsilon/2}(I_n)$, and $J' \subset B_{\varepsilon/2}(I'_n)$ for all n. Using (4.5.3), it follows that

$$B_{\varepsilon/2}(J) \cap B_{\varepsilon/2}(J') \subset B_\varepsilon(I_n) \cap B_\varepsilon(I_n') = \varnothing.$$

Thus setting $I = \overline{N_{\varepsilon/2}(J)}$ and $I' = \overline{B_{\varepsilon/2}(J')}$, we see that I and I' are disjoint compact intervals with $I_n \subset I$ and $I_n' \subset I'$ for all n. That is, (4.5.1) holds.

(2) $\Rightarrow$ (3). Let $x_0 \in \ker P$, I and I' be disjoint compact intervals in $\mathbf{R}$, and let $\{x_n\}$ and $\{x_n'\}$ be disjoint sequences with $x_n \to x_0$ and $x_n' \to x_0$ such that

$$P(x_n) \subset I y_1 \qquad \text{and} \qquad P(x_n') \subset I' y_1$$

for all n. By translating x_0, x_n, and x'_n by an appropriate multiple of y_1, we may assume that

$$\sup I < -\delta < \delta < \inf I'$$

for some $\delta > 0$. Further, by scaling these same vectors by δ^{-1}, we may also assume that

$$\sup I < -1 < 1 < \inf I'. \tag{4.5.4}$$

Using the well-known upper semicontinuity of P, it follows that for each $\varepsilon > 0$,

$$P(x_n) \subset B_\varepsilon(P(x_0)) \qquad \text{and} \qquad P(x'_n) \subset B_\varepsilon(P(x_0))$$

for n sufficiently large. Using (4.5.4) we deduce that $P(x_0) \supset [-1, 1]\, y_1$ or, from the equivalence of (4.1.2) and (4.1.3),

$$\|x_0\| = \|x_0 + y_1\| = \|x_0 - y_0\|. \tag{4.5.5}$$

Now $P(x_n) = [\alpha_n, \beta_n]\, y_1$ for some $\alpha_n \leqslant \beta_n$ and $[\alpha_n, \beta_n] \subset I$ since $P(x_n) \subset I y_1$. By (4.5.4), $\beta_n < -1$ so by Lemma 4.3(2), $J(x_n + y_1) \subset \{x^* \in X_1^* \mid x^*(y_1) < 0\}$. Select any $x_n^* \in \operatorname{ext} J(x_n + y_1)$. Then $x_n^*(y_1) < 0$. By the weak* compactness of X_1^*, there is a subnet of $\{x_n^*\}$ (which we also denote by $\{x_n^*\}$) that weak* converges to some $x_0^* \in X_1^*$. We will show that $x_0^* \in y_1^\perp \cap J(x_0)$. First note that since $x_n^*(y_1) < 0$ for all n, $x_0^*(y_1) \leqslant 0$. If $x_0^*(y_1) < 0$, then there exists $\delta < 0$ such that $x_n^*(y_1) \leqslant -\delta$ eventually. Hence

$$\begin{aligned} \|x_n + y_1\| = x_n^*(x_n + y_1) &\leqslant x_n^*(x_n) - \delta \qquad \text{eventually} \\ &\leqslant \|x_n\| - \delta. \end{aligned}$$

Passing to the limit yields

$$\|x_0 + y_1\| \leqslant \|x_0\| - \delta < \|x_0\|.$$

But this contradicts (4.5.5). Thus we must have $x_0^*(y_1) = 0$ or $x_0^* \in y_1^\perp$.

Also,

$$\begin{aligned} x_0^*(x_0) &= (x_0^* - x_n^*)(x_0) + x_n^*(x_0 - x_n) + x_n^*(x_n + y_1) - x_n^*(y_1) \\ &= (x_0^* - x_n^*)(x_0) + x_n^*(x_0 - x_n) + \|x_n + y_1\| - x_n^*(y_1) \\ &\to 0 + 0 + \|x_0 + y_1\| + 0 = \|x_0\|. \end{aligned}$$

That is, $x_0^* \in J(x_0)$. This proves that $x_0^* \in y_1^\perp \cap J(x_0)$ and $(x_n, x_n^*) \to (x_0, x_0^*)$. Thus $A(x_0, +1) \neq \varnothing$.

A similar proof (using $P(x'_n)$ instead of $P(x_n)$) shows that $A(x_0, -1) \neq \emptyset$. Thus (3) is verified.

(3)$\Rightarrow$(1). Assume that there exists $x_0 \in X$ with $\|x_0\| = \|x_0 - y_1\| = \|x_0 + y_1\|$ such that $A(x_0 + 1) \neq \emptyset$ and $A(x_0, -1) \neq \emptyset$. Let $x_0^* \in A(x_0, +1)$. Then $x_0^* \in y_1^{\perp} \cap J(x_0)$ and there exists a net $(x_n, x_n^*) \in X \times X^*$ such that $x_n \to x_0$, $x_n^* \to x_0^*$ weak*, $x_n^* \in \text{ext}\, J(x_n + y_1)$, and $x_n^*(y_1) < 0$ for all n. Then by Corollary 4.4(2), we have that $\beta_n \leqslant -1$, where $P(x_n) := [\alpha_n, \beta_n]\, y_1$.

Similarly, using the fact that $A(x_0, -1) \neq \emptyset$, we deduce from 4.4(1) that there exists a sequence $\{x'_n\}$ converging to x_0 and $1 \leqslant \alpha'_n$, where $P(x'_n) := [\alpha'_n, \beta'_n]\, y_1$. Clearly, P is not 2-l.s.c. at x_0. Hence P cannot have a continuous selection by Theorem 4.1. ∎

5. One-Dimensional Subspaces in $C_0(T)$

Let T be a locally compact Hausdorff space and let $C_0(T)$ be the linear space of all continuous real functions x on T which "vanish at infinity," i.e., the set $\{t \in T \mid |x(t)| \geqslant \varepsilon\}$ is compact for each $\varepsilon > 0$. Endowed with the supremum norm $\|x\| = \sup\{|x(t)| \mid t \in T\}$, $C_0(T)$ is a Banach space. When T is compact, $C_0(T)$ reduces to the Banach space of all continuous real functions on T, and is usually denoted by $C(T)$.

The *boundary* (resp. *cardinality*) of a subset Z of T is denoted by bd Z (resp. card Z). The *zero set* of an element $x \in C_0(T)$ is the set $Z(x) := x^{-1}(0)$. We shall say that x *does not change* sign at t if there is a neighborhood U of t such that either $x \geqslant 0$ on U or $x \leqslant 0$ on U.

By the well-known Arens–Kelley theorem [1] (stated for T compact but also valid when T is locally compact; see [3, Lemma 3.3] for this and other generalizations),

$$\text{ext}\, C_0(T)_1^* = \{\sigma e_t \mid \sigma = \pm 1,\, t \in T\},$$

where e_t is defined on $C_0(T)$ by

$$e_t(x) := x(t), \qquad x \in C_0(T).$$

Moreover, ext $C_0(T)_1^*$ is weak* closed if T is compact, and when T is not compact, the weak* closure of ext $C_0(T)_1^*$ is the union of ext $C_0(T)_1^*$ and 0.

The main result of this section is the following *intrinsic* characterization of those one-dimensional subspaces in $C_0(T)$ whose metric projections admit continuous selections. In the particular case when T is compact, it had been established by Lazar, Morris, and Wulbert [10] by a rather lengthy ad hoc argument. Our proof is based on Theorem 4.5 and is relatively short and direct.

5.1. THEOREM. *Let* $y_1 \in C_0(T)\backslash\{0\}$. *The following statements are equivalent.*

(1) $P_{[y_1]}$ *has a continuous selection*;

(2) (i) card[bd $Z(y_1)$] $\leqslant 1$ *and*, (ii) *for each* $t \in$ bd $Z(y_1)$, y_1 *does not change sign at* t.

Proof. We may assume $\|y_1\| < 1$.

(1) $\Rightarrow$ (2). Supose (2) fails. Then either card[bd $Z(y_1)$] > 1 or bd $Z(y_1) = \{t_0\}$ and y_1 changes sign at t_0.

Case 1. card[bd $Z(y_1)$] > 1.

Choose distinct points t_0 and t_0' in bd $Z(y_1)$. There are four possible locations for these points: either (i) $t_0, t_0' \in \overline{y_1^{-1}((0, \infty))}$, (ii) $t_0 \in \overline{y_1^{-1}((0, \infty))}$ and $t_0' \in \overline{y_1^{-1}((-\infty, 0))}$, (iii) $t_0, t_0' \in \overline{y_1^{-1}((-\infty, 0))}$, or (iv) $t_0 \in \overline{y_1^{-1}((-\infty, 0))}$ and $t_0' \in \overline{y_1^{-1}((0, \infty))}$. By replacing y_1 with $-y_1$, (iii) is subsumed in (i) and (iv) is subsumed in (ii). Thus we need only consider the possibilities (i) and (ii).

Case 1(i). $t_0, t_0' \in \overline{\{t \in T \mid y_1(t) > 0\}}$.

Choose disjoint neighborhoods U_0 to t_0 and U_0' of t_0'. Using Urysohn's lemma, it is possible to choose an $x_0 \in C_0(T)$ such that $0 \geqslant x_0 \geqslant (-1 + y_1) \wedge (-1 - y_1)$ on U_0, $0 \leqslant x_0 \leqslant (1 + y_1) \wedge (1 - y_1)$ on U_0', $x_0 = 0$ off $U_0 \cup U_0'$, $x_0(t_0) = -1$, and $x_0(t_0') = 1$. Then $\|x_0\| = 1$. Also, for $t \in U_0$,

$$\begin{aligned} -1 &\leqslant y_1(t) \leqslant y_1(t) - x_0(t) \leqslant y_1(t) - [(-1 + y_1(t)) \vee (-1 - y_1(t))] \\ &= y_1(t) + (1 - y_1(t)) \wedge (1 + y_1(t)) \\ &= 1 \wedge (1 + 2y_1(t)) \leqslant 1. \end{aligned}$$

For $t \in U_0'$,

$$-1 \leqslant -y_1(t) \leqslant x_0(t) - y_1(t) \leqslant 1 \wedge (1 - 2y_1(t)) \leqslant 1.$$

For $t \notin U_0 \cup U_0'$,

$$|x_0(t) - y_1(t)| = |y_1(t)| \leqslant 1.$$

Finally, since $|x_0(t_0) - y_1(t_0)| = |x_0(t_0)| = 1$, it follows that $\|x_0 - y_1\| = 1$. Similarly, $\|x_0 + y_1\| = 1$. Thus $\|x_0\| = \|x_0 - y_1\| = \|x_0 + y_1\|$.

Let $\mathscr{U}$ be the collection of all neighborhoods U of t_0, ordered by inclusion. Then $\bigcap \{U \mid U \in \mathscr{U}\} = \{t_0\}$ and, for each $U \in \mathscr{U}$, we can choose a point $t_U \in U$ such that $y_1(t_U) > 0$. Then the net $\{t_U\}$ converges to t_0. By Urysohn's lemma, for each $U \in \mathscr{U}$, there exists a function $g_U \in C_0(T)$ such that $0 \leqslant g_U \leqslant 1$, $g_U = 0$ off U, and $g_U(t_U) = 1$. Then the function $x_U :=$

$g_U(-1-x_0-y_1)+x_0$ is in $C_0(T)$, $x_U \leqslant x_0$ since $-1-(x_0+y_1) \leqslant 0$, and $x_U = x_0$ off U. Given any $\varepsilon > 0$, choose a neighborhood U_ε of t_0 such that $x_0(t) < -1+\varepsilon/2$ and $y_1(t) < \varepsilon/2$ for $t \in U_\varepsilon$. Thus for every $U \in \mathscr{U}$ with $U \subset U_\varepsilon$, and for all $t \in U$, we see that

$$|x_U(t) - x_0(t)| \leqslant y_1(t) + x_0(t) + 1 < \varepsilon.$$

Hence $\|x_U - x_0\| < \varepsilon$ so the net $\{x_U\}$ converges to x_0. Further, for every $t \in T$,

$$\begin{aligned}|x_U(t) + y_1(t)| &= |g_U(t)(-1) + [1-g_U(t)]\{x_0(t) + y_1(t)\}| \\ &\leqslant g_U(t) + [1-g_U(t)]\, \|x_0 + y_1\| = 1\end{aligned}$$

since $\|x_0 + y_1\| = 1$. Since

$$x_U(t_U) + y_1(t_U) = -g_U(t_U) = -1,$$

it follows that $\|x_U + y_1\| = 1$ and

$$-[x_U(t_U) + y_1(t_U)] = \|x_U + y_1\|.$$

Setting $x_U^* = -e_{t_U}$, we see that $x_U^* \in \operatorname{ext} J(x_U + y_1)$, $x_U^*(y_1) < 0$, and $\{x_U^*\}$ converges weak* to $x_0^* := -e_{t_0} \in J(x_0) \cap y_1^\perp$. This proves that $A(x_0, 1) \neq \varnothing$.

Similarly, there exist a net $t'_U \to t'_0$ with $y_1(t'_U) > 0$ for all U and functions $x'_U \in C_0(t)$ with $x'_U \to x_0$ such that

$$x'_U(t'_U) - y_1(t'_U) = \|x'_U - y_1\|.$$

Then the functions $x_U^{*\prime} := e_{t'_U}$ are in $\operatorname{ext} J(x'_U - y_1)$, $x_U^{*\prime}(-y_1) < 0$, and $\{x_U^{*\prime}\}$ converges weak* to $x_0^{*\prime} := e_{t'_0} \in J(x_0) \cap y_1^\perp$. This proves that $A(x_0, -1) \neq \varnothing$.

Case 1(ii). $t_0 \in \overline{\{t \in T \mid y_1(t) > 0\}}$ and $t'_0 \in \overline{\{t \in T \mid y_1(t) < 0\}}$.

Choose neighborhoods U_0 of t_0 and U'_0 of t'_0. By Urysohn's lemma, there exists $x_0 \in C_0(T)$ so that $0 \geqslant x_0 \geqslant (-1+y_1) \vee (-1-y_1)$, $x_0 = 0$ off $U_0 \cup U'_0$, and $x_0(t_0) = x_0(t'_0) = -1$. Then $\|x_0\| = 1$. Furthermore, just as in Case 1(i), it is easy to verify that $\|x_0\| = \|x_0 - y_1\| = \|x_0 + y_1\|$.

Let $\mathscr{U}$ denote the collection of all neighborhoods U of t_0 ordered by inclusion. For each $U \in \mathscr{U}$, choose any $t_U \in U$ such that $y_1(t_U) > 0$. Again, just as in the proof of Case 1(i), we obtain a net $\{x_U\}$ in $C_0(T)$ converging to x_0 and elements $x_U^* = -e_{t_U} \in \operatorname{ext} J(x_U + y_1)$ with $x_U^*(y_1) < 0$ and $\{x_U^*\}$ weak* converges to some $x_0^* = -e_{t_0} \in J(x_0) \cap y_1^\perp$. Thus $A(x_0, 1) \neq \varnothing$.

Next let $\mathscr{U}'$ denote the collection of all neighborhoods U' of t'_0 ordered by inclusion. For each $U' \in \mathscr{U}$, we can choose $t_{U'} \in U'$ such that $y_1(t_{U'}) < 0$. By Urysohn's lemma, choose $g_{U'} \in C_0(T)$ such that $0 \leqslant g_{U'} \leqslant 1$, $g_{U'}(t_{U'}) = 1$, and $g_{U'} = 0$ off U'. Then, arguing just as in Case 1, the function $x_{U'} = g_{U'}[-1-x_0+y_1] + x_0$ is in $C_0(T)$, $x_{U'} \to x_0$, and $x_{U'}(t_{U'}) - y_1(t_{U'}) =$

$-\|x_{U'}-y_1\|$. Thus the functions $x^*_{U''} := -e_{t_{U'}}$ are in $\operatorname{ext} J(x_{U'}-y_1)$ and weak* converge to $x_0^* := -e_{t'_0} \in J(x_0) \cap y_1^\perp$. This proves that $A(x_0, -1) \neq \emptyset$.

Case 2. $\operatorname{bd} Z(y_1 = \{t_0\}$ and y_1 changes sign at t_0, i.e., $t_0 \in \overline{\{t \in T \mid y_1(t) > 0\}} \cap \overline{\{t \in T \mid y_1(t) < 0\}}$.

Let U_0 be a neighborhood of t_0 and set $U'_0 = U_0$. Then by the *exact same proof* as in Case 1(ii), we obtain $x_0 \in C_0(T)$ with $\|x_0\| = \|x_0 - y_1\| = \|x_0 + y_1\|$, $A(x_0, 1) \neq \emptyset$, and $A(x_0, 1) \neq \emptyset$.

In every case we have obtained an $x_0 \in C_0(T)$ with $\|x_0\| = \|x_0 - y_1\| = \|x_0 + y_1\|$, $A(x_0, +1) \neq \emptyset$, and $A(x_0, -1) \neq \emptyset$. By Theorem 4.5, $P_{[y_1]}$ does not have a continuous selection. That is, (1) fails.

(2)$\Rightarrow$(1). Suppose (1) fails. Then by Theorem 4.5, there exists $x_0 \in C_0(T)$ with $\|x_0\| = \|x_0 - y_1\| = \|x_0 + y_1\|$ such that $A(x_0, +1) \neq \emptyset$ and $A(x_0, -1) \neq \emptyset$. Since $A(x_0, +1) \neq \emptyset$, there exist $x_0^* \in J(x_0) \cap y_1^\perp$ and a net $(x_n, \sigma_n e_{t_n})$ in $C_0(T) \times C_0(T)^*$ with $x_n \to x_0$, $\sigma_n = \pm 1$, $\sigma_n e_{t_n} \in J(x_n + y_1)$, $\sigma_n e_{t_n} \to x_0^*$ weak*, and $\sigma_n e_{t_n}(y_1) < 0$ for all n. Since $x_0^* \neq 0$, it must be an extreme point so $x_0^* = \sigma_0 e_{t_0}$ for some $\sigma_0 = \pm 1$, $t_0 \in T$. Since $\sigma_n e_{t_n}$ weak* converges to $\sigma_0 e_{t_0}$, we must have $\sigma_n \to \sigma_0$ and $t_n \to t_0$. By passing to a subnet, we may assume that $\sigma_n = \sigma_0$ for all n. Since $x_1^* \in y_1^\perp$ and $\sigma_0 y_1(t_n) < 0$ for all n, it follows that $t_0 \in \operatorname{bd} Z(y_1)$. Since $x_0^* \in J(x_0)$, we have $\sigma_0 x_0(t_0) = \|x_0\|$ so $\sigma_0 = \operatorname{sgn} x_0(t_0)$.

Similarly, since $A(x_0, -1) \neq \emptyset$, there exist $x_0^{*\prime} \in J(x_0) \cap y_1^\perp$ and a net $(x'_n, \sigma_n e_{t'_n})$ in $C_0(T) \times C_0(T)^*$ with $x'_n \to x_0$, $\sigma_n = \pm 1$, $\sigma_n e_{t'_n} \to x_0^{*\prime}$ weak*, $\sigma_n e_{t'_n} \in J(x'_n - y_1)$, and $\sigma_n e_{t'_n}(-y_1) < 0$ for all n. Arguing as above, we deduce that $x_0^{*\prime} = \sigma'_0 e_{t'_0}$, $\sigma'_0 = \operatorname{sgn} x_0(t'_0)$, $t'_0 \in \operatorname{bd} Z(y_1)$, $\sigma'_0 y_1(t'_n) > 0$ eventually, and $t'_n \to t'_0$. If $t_0 \neq t'_0$, then condition (2)(i) is violated. If $t_0 = t'_0$, then $\sigma_0 = \sigma'_0$ and $y_1(t_n) y_1(t'_n) < 0$ so y_1 changes sign at t_0. Thus condition (2)(ii) is violated. This proves that (2) fails. ∎

It would be of interest to know whether an analogous result holds for *any* finite-dimensional subspace in $C_0(T)$.* In the special case when $T = [a, b]$, this has been accomplished as the culmination of a long series of papers by Nürnberger and Sommer (see their survey paper [12] and the references cited there).

When T is any set with the discrete topology, the boundary of any subset of T is empty and condition (2) of Theorem 5.1 is vacuously satisfied. In particular, if T is the set of natural numbers with the discrete topology, then $C_0(T) =: c_0$ and we obtain the following corollary.

5.2. COROLLARY. *The metric projection onto every one-dimensional subspace in c_0 has a continuous selection.*

* Wu Li has recently produced such a result in the case when T is locally connected.

Actually, as a consequence of a result of Blatter [2], c_0 has the property (P) of Brown [4]. Thus the metric projection onto *any* finite-dimensional subspace of c_0 is l.s.c. and hence has a continuous selection.

6. One-Dimensional Subspace in L_1

Let $(T, \mathscr{S}, \mu)$ be a measure space and let $L_1 = L_1(T, \mathscr{S}, \mu)$ denote the Banach space of all integrable functions x on T with the norm

$$\|x\| = \int_T |x| \, d\mu.$$

The *support* of a function $x \in L_1$ is the set $\operatorname{supp} x := \{t \in T \mid x(t) \neq 0\}$ and the *zero set* of x is

$$Z(x) := T \backslash \operatorname{supp} x = \{t \in T \mid x(t) = 0\}.$$

(Here, and in the sequel, all sets in T are only defined up to a set of μ-measure zero.) We shall assume that $L_1^* = L_\infty$. This will be the case, for example, if $(T, \mathscr{S}, \mu)$ is σ-finite.

A set $A \in \mathscr{S}$ is called an *atom* if $\mu(A) > 0$ and either $\mu(B) = 0$ or $\mu(B) = \mu(A)$ whenever $B \in \mathscr{S}$ and $B \subset A$.

The following lemma collects some facts about atoms that will be needed in this section.

6.1. Lemma. (1) *For each* $x \in L_1$, supp(x) *is "σ-finite," i.e., is a countable union of sets having finite measure.*

(2) *There are at most a countable number of atoms in a σ-finite set, and each such atom has finite measure.*

(3) *A measurable function x is constant a.e. (μ) on an atom A of finite measure; this value will be denoted by $x(A)$.*

(4) *If $E \in \mathscr{S}$ has the property that $0 < \mu(E) \leqslant \infty$ and E contains no atoms, then for each sequence of positive numbers $\{\varepsilon_n\}$ there exists a sequence of pairwise disjoint sets $\{E_n\}$ in E with $0 < \mu(E_n) < \varepsilon_n$ for every n.*

These facts seem fairly well known with the possible exception of (4). However, (4) can be proved using the same idea as in the proof of Sak's lemma [6, pp. 308–309].

6.2. Definitions. A set $E \in \mathscr{S}$ will be called *unifat* if it is the union of a finite number of atoms. An element $y_1 \in L_1$ is said to satisfy the *Lazar condition* if whenever A and B are disjoint sets in $\mathscr{S}$ with $A \cup B = \operatorname{supp}(y_1)$ and $\int_A |y_1| \, d\mu = \int_B |y_1| \, d\mu$, then one of the sets A or B must be unifat.

The main result of this section is the following *intrinsic* characterization of those one-dimensional subspaces in L_1 whose metric projections admit continuous selections.

6.3. THEOREM. *Let $y_1 \in L_1 \backslash \{0\}$. Then $P_{[y_1]}$ has a continuous selection if and only if y_1 satisfies the Lazar condition.*

Before giving the proof, we observe a few immediate corollaries of this theorem.

6.4. COROLLARY. *If* $\operatorname{supp}(y_1)$ *contains no atoms, then $P_{[y_1]}$ does not have a continuous selection.*

Related to this corollary, Lazar, Morris, and Wulbert [10] proved that if T contained no atoms, then the metric projection onto any finite-dimensional subspace in $L_1(T, \mathcal{S}, \mu)$ does not have a continuous selection.

6.5. COROLLARY. *If* $\operatorname{supp}(y_1)$ *is a finite union of atoms, then $P_{[y_1]}$ has a continuous selection. In particular, if L_1 is finite dimensional, the metric projection onto any one-dimensional subspace has a continuous selection.*

The next corollary is due to Lazar and provided the motivation for the name "Lazar condition." It follows immediately from Theorem 6.3 taking $T = \mathbf{N} =$ "the natural numbers," $\mathcal{S}$ all subsets of $\mathbf{N}$, and μ counting measure on $\mathcal{S}$, i.e., $\mu(E) = \operatorname{card}(E)$. Then $L_1(T, \mathcal{S}, \mu) = l_1$ and Theorem 6.3 reduces to

6.6. COROLLARY (Lazar [9]). *Let $y_1 \in l_1 \backslash \{0\}$. Then $P_{[y_1]}$ has a continuous selection if and only if whenever A and B are disjoint sets of integers with*

$$A \cup B = \{i \in \mathbf{N} \mid y_1(i) \neq 0\}$$

and

$$\sum_{i \in A} |y_1(i)| = \sum_{i \in B} |y_1(i)|,$$

then either A or B must be finite.

The remainder of this section is devoted to a proof of Theorem 6.3.

6.7. LEMMA. *Let T_1 be a σ-finite subset of T. If T_1 is not unifat, then there exists a sequence $\{E_n\}$ of pairwise disjoint subsets of T_1 with $0 < \mu(E_n) < \infty$ such that, for any $x \in L_1$,*

$$\lim_{n \to \infty} \int_{E_n} |x| \, d\mu = 0. \tag{6.7.1}$$

Proof. Since T_1 is not unifat, either T_1 is the union of a countable infinity of atoms, or $\mu(T_1\backslash A)>0$, where A is the union of all atoms in T_1. In the latter case, we set $E=T_1\backslash A$ and apply Lemma 6.1(4). This yields a sequence $\{E_n\}$ of pairwise disjoint subsets of T with $\lim_{n\to\infty}\mu(E_n)=0$. For any $x\in L_1$, the condition (6.7.1) is a well-known consequence of the monotone convergence theorem.

In the former case, we have

$$T_1=\bigcup_1^\infty A_n,$$

where $\{A_n\}$ is a sequence of pairwise disjoint atoms. Now for each $x\in L_1$,

$$\sum_1^\infty |x(A_n)|\,\mu(A_n)=\int_{\bigcup_1^\infty A_n}|x|\,d\mu\leqslant\int_T |x|\,d\mu=\|x\|<\infty$$

so $\lim_{n\to\infty}|x(A_n)|\,\mu(A_n)=0$. Hence

$$\int_{A_n}|x|\,d\mu=|x(A_n)|\,\mu(A_n)\to 0.$$

Setting $E_n=A_n$ for all n, the result follows. ∎

It is well known that $\text{ext}(L_\infty)$ consists of all those measurable functions f on T such that $|f|=1$ a.e. (μ). Also, for each $x^*\in L_1^*$, there is a unique $f\in L_\infty$ such that

$$x^*(x)=\int_T xf\,d\mu,\qquad x\in L_1,$$

and $\|x^*\|=\text{ess supp}\,|f|$. We shall call f the *representer* of x^*.

6.8. Lemma. *Let $y_1\in L_1\backslash\{0\}$ and let $x_0\in L_1$ satisfy*

$$\|x_0\|=\|x_0+y_1\|=\|x_0-y_1\|. \tag{6.8.1}$$

Let $x_0^*\in y_1^\perp\cap J(x_0)$, suppose $f_0\in L_\infty$ is the representer of x_0^*, and define

$$T^+:=\{t\in\text{supp}\,y_1\,|\,f_0(t)=\text{sgn}\,y_1(t)\}$$
$$T^-:=\{t\in\text{supp}\,y_1\,|\,f_0(t)=-\text{sgn}\,y_1(t)\}.$$

Then:

(1) $\text{supp}\,f_0\supset\text{supp}\,x_0\supset\text{supp}\,y_1$ *and* $\text{supp}\,y_1=T^+\cup T^-$.

(2) $T^+\subset\text{supp}(x_0+y_1)$, $T^-\subset\text{supp}(x_0-y_1)$.

(3) *If* T^+ *is unifat, then* $x_0^* \notin A(x_0, 1)$; *if* T^- *is unifat, then* $x_0^* \notin A(x_0, -1)$.

Proof. (1) By (6.8.1), it follows that $x_0^* \in J(x_0 - y_1) \cap J(x_0 + y_1)$. Thus

$$\int_T x_0 f_0 \, d\mu = \|x_0\| \qquad \text{and} \qquad \int_T (x_0 \pm y_1) f_0 \, d\mu = \|x_0 \pm y_1\|$$

implies that

$$f_0(t) = \begin{cases} \text{sign } x_0(t) & \text{a.e. on supp } x_0 \\ \text{sgn}[x_0(t) + y_1(t)] & \text{a.e. on supp}(x_0 + y_1) \\ \text{sgn}[x_0(t) - y_1(t)] & \text{a.e. on supp}(x_0 - y_1). \end{cases} \tag{6.8.2}$$

Hence if $t \in \text{supp } y_1 \backslash \text{supp } x_0$, we have that

$$f_0(t) = \text{sgn } y_1(t) = -\text{sgn } y_1(t),$$

which is impossible. Hence

$$\mu(\text{supp } y_1 \backslash \text{supp } x_0) = 0,$$

or supp $y_1 \subset$ supp x_0. Also, since $f_0(t) = \text{sgn } x_0(t)$ on supp x_0, it follows that supp $x_0 \subset$ supp f_0. Finally, if $t \in \text{supp } y_1$, then $f_0(t) = \pm 1$ so supp $y_1 = T^+ \cup T^-$.

(2) If $t \in T^+$, then $t \in \text{supp } y_1$ and $f_0(t) = \text{sgn } y_1(t)$. But supp $y_1 \subset$ supp x_0 and $f_0(t) = \text{sgn } x_0(t)$. Thus $y_1(t) x_0(t) > 0$ and $t \in \text{supp}(x_0 + y_1)$. That is, $T^+ \subset \text{supp}(x_0 + y_1)$. Similarly, $T^- \subset \text{supp}(x_0 - y_1)$.

(3) Assume T^+ is unifat. Then $T^+ = \bigcup_1^k A_i$, where the A_i are pairwise disjoint atoms and $0 < \mu(A_i) < \infty$. To show that $x_0^* \notin A(x_0, 1)$, it suffices to show that if $\{(x_n, x_n^*)\}$ is any net in $L_1 \times L_1^*$ with $x_n \to x_0$, $x_n^* \to x_0^*$ weak*, and $x_n^* \in J(x_n + y_1)$, then $x_n^*(y_1) \geqslant 0$ eventually. Let $\{(x_n, x_n^*)\}$ be such a net and let $f_n \in L_\infty$ be the representer of x_n^*. Since $T^+ \subset \text{supp}(x_0 + y_1)$ by (2), it follows that $x_0(A_i) + y_1(A_i) \neq 0$ for $i = 1, 2, ..., k$. Since $x_n + y_1 \to x_0 + y_1$, it follows that $x_n(A_i) + y_1(A_i) \to x_0(A_i) + y_1(A_i)$ so $x_n(A_i) + y_1(A_i) \neq 0$ eventually (for $i = 1, 2, ..., k$). Since $x_n^* \in J(x_n + y_1)$,

$$\begin{aligned} f_n(A_i) &= \text{sgn}[x_n(A_i) + y_1(A_i)] \\ &= \text{sign}[x_0(A_i) + y_1(A_i)] && \text{eventually} \\ &= f_0(A_i) && \text{by (6.8.2).} \end{aligned}$$

That is, $f_n = f_0$ eventually on T^+. Hence, for n eventually,

$$
\begin{aligned}
x_n^*(y_1) &= \int_T f_n y_1 \, d\mu = \int_{T^+ \cup T^-} f_n y_1 \, d\mu \\
&= \int_{T^+} f_0 y_1 \, d\mu + \int_{T^-} f_n y_1 \, d\mu \\
&\geqslant \int_{T^+} f_0 y_1 \, d\mu - \int_{T^-} |y_1| \, d\mu = \int_{T^+} f_0 y_1 \, d\mu + \int_{T^-} f_0 y_1 \, d\mu \\
&= \int_{T^+ \cup T^-} f_0 y_1 \, d\mu = x_0^*(y_1) = 0.
\end{aligned}
$$

Thus $x_0^* \notin A(x_0, 1)$.

Similarly, if T^- is unifat, then $x_0^* \notin A(x_0, -1)$. ▌

Now we are in position to prove Theorem 6.3.

Proof of Theorem 6.3. Suppose y_1 fails the Lazar condition. We will show that there exists $x_0 \in L_1$ with $\|x_0\| = \|x_0 - y_1\| = \|x_0 + y_1\|$ such that $A(x_0, 1) \neq \varnothing$ and $A(x_0, -1) \neq \varnothing$. The assumption on y_1 implies that there exist disjoint sets A and B such that $A \cup B = \operatorname{supp} y_1$, $\int_A |y_1| \, d\mu = \int_B |y_1| \, d\mu$, and neither A nor B is unifat. By Lemma 6.7, there exist sequences $\{E_n\}$ and $\{F_n\}$ of pairwise disjoint sets such that $E_n \subset A$, $F_n \subset B$, $0 < \mu(E_n) < \infty$, $0 < \mu(F_n) < \infty$, and, for each $x \in L_1$,

$$\lim_{n \to \infty} \int_{E_n} |x| \, d\mu = 0 = \lim_{n \to \infty} \int_{F_n} |x| \, d\mu. \tag{6.3.1}$$

Define functions f_0 and x_0 on T by

$$f_0(t) = \begin{cases} \operatorname{sgn} y_1(t) & \text{for } t \in A \\ -\operatorname{sgn} y_1(t) & \text{for } t \in B \\ 1 & \text{otherwise} \end{cases}$$

and

$$x_0(t) = \begin{cases} y_1(t) & \text{for } t \in A \\ -y_1(t) & \text{for } t \in B \\ 0 & \text{otherwise.} \end{cases}$$

Then $f_0 \in L_\infty$, ess sup $|f_0| = 1$, and $x_0 \in L_1$. A simple computation shows that

$$\|x_0\| = \|x_0 - y_1\| = \|x_0 + y_1\| \qquad (= \|y_1\|)$$

and, letting f_0 be the representer of the functional $x_0^* \in L_1^*$, we see that $x_0^* \in y_1^\perp \cap J(x_0)$ and $x_0^* \in \text{ext}(L_1^*)_1$.

Next define functions f_n in L_∞ by

$$f_n = \begin{cases} f_0 & \text{on} \quad T \backslash E_n \\ -f_0 & \text{on} \quad E_n \end{cases}$$

and functions x_n in L_1 by

$$x_n = \begin{cases} x_0 & \text{on} \quad T \backslash E_n \\ -2y_1 & \text{on} \quad E_n. \end{cases}$$

Let f_n be the representer of $x_n^* \in L_1^*$. Then $x_n^* \in \text{ext}(L_1)_1$ and a straightforward computation verifies that

$$x_n^*(x_n + y_1) = \|x_0\| - \int_{E_n} |y_n| \, d\mu = \|x_n + y_1\|.$$

That is, $x_n^* \in \text{ext}\, J(x_n + y_1)$. Next observe that

$$\|x_n - x_0\| = 3 \int_{E_n} |y_1| \, d\mu \to 0$$

by (6.3.1). Further, for any $x \in L_1$,

$$\begin{aligned} x_n^*(x) &= \int_T f_n x \, d\mu = \int_{T \backslash E_n} f_0 x \, d\mu - \int_{E_n} f_0 x \, d\mu \\ &= \int_T f_0 x \, d\mu - 2 \int_{E_n} f_0 x \, d\mu = x_0^*(x) - 2 \int_{E_n} f_0 x \, d\mu \\ &\to x_0^*(x) \qquad \text{by (6.3.1).} \end{aligned}$$

That is, $x_n^* \to x_0^*$ weak*. Finally,

$$\begin{aligned} x_n^*(y_1) &= \int_T f_n y_1 \, d\mu = \int_{T \backslash E_n} f_0 y_1 \, d\mu - \int_{E_n} f_0 y_1 \, d\mu \\ &= \int_T f_0 y_1 \, d\mu - 2 \int_{E_n} f_0 y_1 \, d\mu \\ &= -2 \int_{E_n} |y_1| \, d\mu < 0. \end{aligned}$$

This proves that $x_0^* \in A(x_0, 1)$.

Similarly, defining f'_n and x'_n on T by

$$f'_n = \begin{cases} f_0 & \text{on } T \backslash F_n \\ -f_0 & \text{on } F_n \end{cases} \qquad x'_n = \begin{cases} x_0 & \text{on } T \backslash F_n \\ 2y_1 & \text{on } F_n \end{cases}$$

and letting f'_n be the representers for the functionals $x_n^{*\prime}$, we can prove that $(x'_n, x_n^{*\prime}) \to (x_0, x_0^*)$, $x_n^{*\prime} \in \operatorname{ext} J(x'_n - y_1)$, and $x_n^{*\prime}(-y_1) < 0$ for all n. Thus $x_0^* \in A(x_0, -1)$. By Theorem 4.5, it follows that $P_{[y_1]}$ does not have a continuous selection.

Conversely, suppose $P_{[y_1]}$ fails to have a continuous selection. Then by Theorem 4.5 there exists $x_0 \in L_1$ with

$$\|x_0\| = \|x_0 - y_1\| = \|x_0 + y_1\|$$

such that $A(x_0, +1) \neq \emptyset$ and $A(x_0, -1) \neq \emptyset$. For each $\sigma \in \{+1, -1\}$, let $x_\sigma^* \in A(x_0, \sigma)$ and let $f_\sigma \in L_\infty$ be the representer of x_σ^*. Define

$$T_1(\sigma) := \{t \in \operatorname{supp} y_1 \mid f_\sigma(t) = \operatorname{sgn} y_1(t)\}$$

$$T_2(\sigma) := \{t \in \operatorname{supp} y_1 \mid f_\sigma(t) = -\operatorname{sgn} y_1(t)\}.$$

Since $x_\sigma^* \in y_1^\perp \cap J(x_0)$, Lemma 6.8 implies that

$$\operatorname{supp} f_\sigma \supset \operatorname{supp} x_0 \supset \operatorname{supp} y_1,$$

$$T_1(\sigma) \cup T_2(\sigma) = \operatorname{supp} y_1 \qquad \text{and} \qquad T_1(\sigma) \cap T_2(\sigma) = \emptyset. \tag{6.3.2}$$

Further, $x_\sigma^* \in y_1^\perp$ implies that

$$\int_{T_1(\sigma)} |y_1| \, d\mu = \int_{T_2(\sigma)} |y_1| \, d\mu.$$

If y_1 has the Lazar condition, then for each $\sigma \in \{+1, -1\}$ either $T_1(\sigma)$ or $T_2(\sigma)$ is unifat. If $\sigma = 1$, then since $x_1^* \in A(x_0, +1)$, it follows by Lemma 6.8(3) that $T_1(1)$ is not unifat. Hence $T_2(1)$ is unifat. Similarly, since $x_{-1}^* \in A(x_0, -1)$, $T_2(-1)$ is not unifat and $T_1(-1)$ is unifat. Since $x_\sigma^* \in J(x_0)$ for $\sigma = \pm 1$, it follows that

$$f_1(t) = \operatorname{sgn} x_0(t) = f_{-1}(t)$$

for all $t \in \operatorname{supp} x_0 \supset \operatorname{supp} y_1$. Thus $T_1(1) \cap T_2(-1) = \emptyset$ and $T_1(-1) \cap T_2(1) = \emptyset$. But this implies by (6.3.2) that $T_2(-1) \subset T_2(1)$ and $T_2(1) \subset T_2(-1)$. That is, $T_2(-1) = T_2(1)$. But $T_2(1)$ is unifat and $T_2(-1)$ is not unifat. This contradiction shows that y_1 fails to have the Lazar condition, and completes the proof. ∎

It would be interesting to know whether there is an analogue of Theorem 6.3 for subspaces of L_1 of *any* finite dimension greater than one.

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